Improved Method for Determining Free-Free Modes Using Constrained Test Data

Wan-Zhi Han,* Su-Huan Chen,† and Yu-Guang Guo‡
Jilin University of Technology, Changchun, 130022, People's Republic of China

Introduction

BY using the experimental data of the same structure in a constrained state, Przemieniecki[1] derived an analytical method for extracting the free-free modes of an unconstrained structure. Recently, Chen et al.[2] proposed a method to improve Przemieniecki's method by modal truncation so that only the lower constrained modes need to be measured. To get a more accurate free-free modes, Zhang and Zerva[3] proposed a new method that considers the compensation of the higher modes of the constrained structure.

In this Note, an improved method for determining the free-free modes using the lower test modes of the same structure in constrained state is proposed. The result obtained by the proposed method is more exact than that obtained by Zhang and Zerva's method. Thus, it is a more effective method for the modal analysis of large aerospace structures.

Methods of Przemieniecki, Chen et al., and Zhang and Zerva

The dynamic equation of a freely vibration unconstrained system is

$$\left(\begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{bmatrix}\right)\begin{bmatrix} q_x \\ q_y \end{bmatrix} = 0 \tag{1}$$

$$M_{xx} = K_{xx}\bar{P}_e\bar{\Omega}_e^{-2}\bar{P}_e^{-1} \tag{2}$$

where $\bar{P}_e$ and $\bar{\Omega}_e$ are the test modes of the constrained system.

In Eq. (2), the calculation of M_{xx} requires that all of the frequencies and associated mode shapes of the constrained structure must be measured from the vibration test. To bypass this requirement, Chen et al.[2] partitioned $\bar{\Omega}_e$ and $\bar{P}_e$ into lower and higher frequencies $\bar{\Omega}_{el}$ and $\bar{\Omega}_{eh}$ and mode shapes $\bar{P}_{el}$ and $\bar{P}_{eh}$, respectively. Then, M_{xx} can be expressed as

$$M_{xx} = K_{xx}\bar{P}_{el}\bar{\Omega}_{el}^{-4}\bar{P}_{el}^T K_{xx} + K_{xx}\bar{P}_{eh}\bar{\Omega}_{eh}^{-4}\bar{P}_{eh}^T K_{xx} \tag{3}$$

and an approximate expression of M_{xx} can be obtained by modal truncation under the assumption $\bar{\Omega}_{eh}^{-4} \ll \bar{\Omega}_{el}^{-4}$,

$$M_{xx} \approx K_{xx}\bar{P}_{el}\bar{\Omega}_{el}^{-4}\bar{P}_{el}^T K_{xx} \tag{4}$$

where the superscript T represents matrix transpose.

Actually, the separation between frequencies described in Ref. 2 is not usually observed in large space structures. Thus, Eq. (4) is an incomplete estimate for the mass submatrix M_{xx}, and the contribution of the higher modes to M_{xx} should not be neglected. By normalizing the mode shapes $\bar{P}_e$ with the stiffness matrix K_{xx} and using modes partition technique, M_{xx} can be expressed as

$$M_{xx} = K_{xx}\bar{P}_{el}\bar{\Omega}_{el}^{-2}\bar{P}_{el}^T K_{xx} + (I - K_{xx}\bar{P}_{el}\bar{P}_{el}^T)M_{xx}(I - \bar{P}_{el}\bar{P}_{el}^T K_{xx}) \tag{5}$$

where I represents the unit matrix.

In Eq. (5), the second term in the right-hand side is the contribution of the higher modes to the mass matrix M_{xx}. By replacing the mass matrix M_{xx} in the right-hand side of Eq. (5) with the analytically evaluated mass matrix M_{axx}, Zhang and Zerva[3] gave the following approximate expression of M_{xx}:

$$M_{xx} \approx K_{xx}\bar{P}_{el}\bar{\Omega}_{el}^{-2}\bar{P}_{el}^T K_{xx} + (I - K_{xx}\bar{P}_{el}\bar{P}_{el}^T)M_{axx}(I - \bar{P}_{el}\bar{P}_{el}^T K_{xx}) \tag{6}$$

Improved Method for the Compensation of the Modal Truncation

If the analytically evaluated mass matrix M_{axx} is updated by the known lower test modes of the constrained structure, then we can get a more exact expression of M_{xx} than that obtained by Zhang and Zerva's method.[3]

Assume P_{al} and Ω_{al} are the lower analytical mode shapes and frequencies obtained by solving the following eigenproblem:

$$K_{xx}P_{al} = M_{axx}P_{al}\Omega_{al}^2 \tag{7}$$

where K_{xx} is the stiffness matrix obtained from the static test of the constrained structure and M_{axx} is the mass matrix obtained from the finite element analysis of the constrained structure. The lower test mode shapes $\bar{P}_{el}$ and frequencies $\bar{\Omega}_{el}$ satisfy

$$K_{xx}\bar{P}_{el} = M_{xx}\bar{P}_{el}\bar{\Omega}_{el}^2 \tag{8}$$

where K_{xx} is the same as in Eq. (7) and M_{xx} is the real mass matrix, which can not be obtained by experimental method. So there is an error between M_{axx} and M_{xx}, assume

$$M_{xx} = M_{axx} + \Delta M \tag{9}$$

$$\bar{P}_{el} = P_{al} + \Delta P_l \tag{10}$$

$$\bar{\Omega}_{el}^2 = \Omega_{al}^2 + \Delta\Lambda_l \tag{11}$$

By substituting Eqs. (9–11) into Eq. (8), we have

$$\Delta M\bar{P}_{el}\bar{\Omega}_{el}^2 = \chi \tag{12}$$

where $\chi = K_{xx}\Delta P_l - M_{xx}(\Delta P_l\bar{\Omega}_{el}^2 + P_{al}\Delta\Lambda_l)$

Rewrite Eq. (12) in the following form:

$$\boldsymbol{AX} = \boldsymbol{B} \tag{13}$$

where vector $\boldsymbol{X}$ represents the unknown nonzero elements of ΔM corresponding to the nonzero elements of M_{axx}; matrix A is an $m \times n$ matrix, $m = L \times N$; L is the number of the available lower test modes; N is the total degrees of freedom of the system; $\boldsymbol{B}$ is a $(N \times L)$ vector; and the variables of $\boldsymbol{B}$ are the elements of χ.

Because the number of lower test modes L is far less than the number of the structural degrees of freedom, there are more unknown quantities in Eq. (13) than number of equations in Eq. (12); therefore, Eq. (12) is an uncertain set of equations, and only the generalized inverse solution can be obtained. Substituting the solution of Eq. (13), i.e., ΔM, into Eq. (9), we can obtain an improved mass matrix M_{lxx}', which is the updated mass matrix of M_{axx} by using the available lower test modes and closer to the structural real mass matrix M_{xx} than the analytically evaluated mass matrix M_{axx}. Then,

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*Ph.D. Student, Department of Mechanics, Open Laboratory of CAD/CAM, Technology for Advanced Manufacturing Academia Sinica.

†Mechanics Professor, Department of Mechanics, Open Laboratory of CAD/CAM, Technology for Advanced Manufacturing Academia Sinica.

‡Lecturer, Department of Mechanics, Open Laboratory of CAD/CAM, Technology for Advanced Manufacturing Academia Sinica.

Table 1 Comparison of the extracted frequencies obtained by the three methods

	Results					
Order	1st	2nd	3rd	4th	5th	6th
	Exact solution					
	13.7347	13.7347	37.8436	37.8436	74.2851	74.2851
	Frequencies (errors[a]) obtained by proposed method					
m[b] $= 1$	13.7491					
	(0.1050)					
$m = 2$	13.7491	13.7491				
	(0.1050)	(0.1050)				
$m = 3$	13.7330	13.7491	37.9300			
	(−0.0122)	(0.1050)	(0.2274)			
$m = 4$	13.7330	13.7330	37.9308	37.9308		
	(−0.0122)	(−0.0122)	(0.2305)	(0.2305)		
$m = 5$	13.7284	13.7330	37.8365	37.9308	74.0301	
	(−0.0456)	(−0.0122)	(−0.0188)	(0.2305)	(−0.3433)	
$m = 6$	13.7284	13.7284	37.8365	37.8365	74.0301	74.0301
	(−0.0456)	(−0.0456)	(−0.0188)	(−0.0188)	(−0.3433)	(−0.3433)
	Frequencies (errors) obtained by Zhang and Zerva's method[3]					
$m = 1$	14.2940					
	(4.0723)					
$m = 2$	14.2940	14.2940				
	(4.0723)	(4.0723)				
$m = 3$	13.8151	14.2940	39.5808			
	(0.5854)	(4.0723)	(4.5904)			
$m = 4$	13.8151	13.8151	39.5808	39.5808		
	(0.5854)	(0.5854)	(4.5904)	(4.5904)		
$m = 5$	13.7801	13.8151	37.8972	39.5808	77.9549	
	(0.3304)	(0.5854)	(0.1416)	(4.5904)	(4.9400)	
$m = 6$	13.7801	13.7801	37.8972	37.8972	77.9549	77.9549
	(0.3304)	(0.3304)	(0.1416)	(0.1416)	(4.9400)	(4.9400)
	Frequencies (errors) obtained by the Chen et al. method[2]					
$m = 1$	217.3412					
	(1482.4271)					
$m = 2$	217.3412	217.3412				
	(1482.4271)	(1482.4271)				
$m = 3$	14.8348	217.3412	291.7449			
	(8.0097)	(1482.4271)	(670.9225)			
$m = 4$	14.8348	14.8348	291.7449	291.7449		
	(8.0097)	(8.0097)	(670.9225)	(670.9225)		
$m = 5$	14.2482	14.8348	38.4092	291.7449	354.8157	
	(3.7388)	(8.0097)	(1.4946)	(670.9225)	(377.6402)	
$m = 6$	14.2482	14.2482	38.4092	38.4092	354.8157	354.8157
	(3.7388)	(3.7388)	(1.4946)	(1.4946)	(377.6402)	(377.6402)

[a]Defined by $100 \times (\hat{\omega}_i - \omega_i)/\omega_i$, where $\hat{\omega}_i$ is the ith extracted natural frequency.
[b]Number of constrained test modes used in the calculation of free-free modes.

replacing the M_{xx} on the right-hand side of Eq. (5) by M^{l}_{xx}, we can obtain an approximation of M_{xx}:

$$M_{xx} \approx K_{xx}\bar{P}_{el}\Omega_{el}^{-2}\bar{P}_{el}^{T}K_{xx} + \left(I - K_{xx}\bar{P}_{el}\bar{P}_{el}^{T}\right)M^{l}_{xx}\left(I - \bar{P}_{el}\bar{P}_{el}^{T}K_{xx}\right) \tag{14}$$

Because M^{l}_{xx} used in Eq. (14) is closer to the real mass matrix M_{xx} than M_{axx} used in Eq. (6), the approximate value of the mass matrix M_{xx} obtained from Eq. (14) is also closer to the real mass matrix M_{xx} than that obtained from Eq. (6). Therefore, the free-free modes obtained from Eq. (1) by using the mass matrix M_{xx} obtained from Eq. (14) is more exact than that obtained by Zhang and Zerva's method.[3]

Numerical Example

A uniform straight beam (as shown in Fig. 1) is used as an example. The beam is discretized into six elements, and all of the translation and rotation displacements are considered. It is a three-dimensional beam with repeated frequencies. We take the constrained mass matrix obtained by the finite element method when $D = 0.31$ m as the analytically evaluated mass matrix M_{axx} and the constrained stiffness matrix and mass matrix obtained by finite element method when $D = 0.32$ m as the stiffness matrix K_{xx} obtained from the static experiment and the unavailable real mass matrix M_{xx}. We take the lower eigenvectors $\boldsymbol{P}_{al}$ and frequencies Ω_{al} obtained by solving the eigenproblem $K_{xx}\boldsymbol{P}_{al} = M_{axx}\boldsymbol{P}_{al}\Omega_{al}^{2}$ as the analysis modes, and the lower eigenvectors $\boldsymbol{P}_{el}$ and frequencies

Table 2 Comparison of the extracted mode shapes[a] obtained by the three methods

	Results					
Order	1st	2nd	3rd	4th	5th	6th
	Proposed method					
$m = 1$	0.0011					
$m = 2$	0.0011	0.0011				
$m = 3$	0.0003	0.0011	0.0023			
$m = 4$	0.0003	0.0003	0.0023	0.0023		
$m = 5$	0.0013	0.0003	0.0014	0.0023	0.0053	
$m = 6$	0.0013	0.0013	0.0014	0.0014	0.0053	0.0053
	Zhang and Zerva's method[3]					
$m = 1$	0.0455					
$m = 2$	0.0455	0.0455				
$m = 3$	0.0190	0.0455	0.0487			
$m = 4$	0.0190	0.0190	0.0487	0.0487		
$m = 5$	0.0093	0.0190	0.0101	0.0487	0.0489	
$m = 6$	0.0093	0.0093	0.0101	0.0101	0.0489	0.0489
	Chen et al. method[2]					
$m = 1$	11.0681					
$m = 2$	11.0681	11.0681				
$m = 3$	0.2507	11.0681	4.0908			
$m = 4$	0.2507	0.2507	4.0908	4.0908		
$m = 5$	0.1033	0.2507	0.1025	4.0908	2.5327	
$m = 6$	0.1033	0.1033	0.1025	0.1025	3.4907	3.4907

[a]Defined by $\|\hat{P}_i - P_i\| / \|P_i\|$, where $\hat{P}_i$ is the ith extracted mode shape.

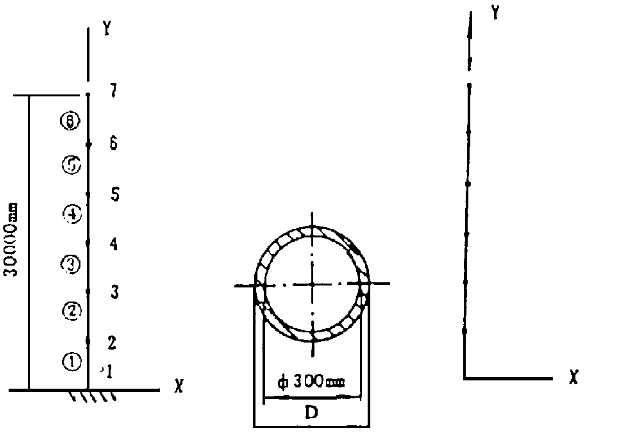

Fig. 1 Experimental data for the clamped-free beam: Young's modulus of elasticity $E = 6.9 \times 10^{10}$ N/m^2, mass density $\rho = 2.7143 \times 10^3$ kg/m^3.

$\bar{\Omega}_{el}$ obtained by solving the eigenproblem $K_{xx}\bar{P}_{el} = M_{xx}\bar{P}_{el}\bar{\Omega}^2_{el}$ as the simulated test modes of the constrained structure. Then we can obtain the updated mass matrix M^u_{xx} from Eqs. (9–11). Next, we can obtain an approximate M_{xx} by substituting M^u_{xx} into the right-hand side of Eq. (14). Finally, solving the eigenproblem defined by Eq. (1) we can obtain the mode shapes and associated frequencies of the unconstrained structure. The results listed in Tables 1 and 2 are the comparison of the first six nonrigid frequencies and associated mode shapes of the unconstrained structure obtained by the proposed method, Zhang and Zerva's method,[3] and the Chen et al. method,[2] respectively. It is clearly seen that both the frequencies and mode shapes obtained by the proposed method are more accurate than that obtained by the other two methods, even for the case that only a few of the constrained modes are available. Furthermore, the proposed method is also suitable for the case when the frequencies of the structure are repeated or closed.

Conclusions

By using the available lower constrained test modes to update the analytical evaluated mass matrix, an improved method for determining the free-free modes is proposed. The proposed method can yield more accurate free-free modes than Zhang and Zerva's method.[3] Because the proposed method is suitable for the structure with repeated or closed frequencies and does not require more test modes of the constrained structure, it is very important for determining the free-free modes of the large aerospace structures.

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